Random matrix theory and L-functions

Zeros of symmetric power L functions

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February 2008, Warwick University

What is a L-function?

1. A Dirichlet series with Euler product of degree at most $d \geq 1$ (and degree $< d$ only for a finite number of factors):

$$L(\varpi, s) = \sum_{n=1}^{+\infty} \lambda_{\varpi}(n) n^{-s}$$

$$= \prod_{p \in \mathscr{P}} \prod_{j=1}^{d} \left[1 - \frac{\alpha_{\varpi,j}(p)}{p^s}\right]^{-1} \quad (\alpha_{\varpi,j}(p) \in \mathbb{C})$$

We assume $\alpha_{\varpi,j}(p) \ll 1$ hence the series and product converge absolutely for $\Re\mathfrak{e}\, s > 1$.

2. A gamma factor:

$$\gamma(\varpi, s) = \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s+\kappa_j}{2}\right)$$

with

$$\kappa_j \in \mathbb{R}_{>-1} \text{ or } \left| \begin{array}{l} \kappa_j \in \mathbb{C} \\ \exists j' : \kappa_{j'} = \overline{\kappa_j} \\ \Re\mathfrak{e}\, \kappa_j > -1. \end{array} \right.$$

hence $\gamma(\varpi, s)$ has no zeros in $\mathbb{C}$ and no pole in $\Re\mathfrak{e}\, s \geq 1$.

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What is a L-function? (cont.)

3. A positive integer $q(\varpi) \geq 1$: the conductor of $L(\varpi, s)$ such that:

$$p \nmid q(\varpi) \Longrightarrow \alpha_{\varpi,j}(p) \neq 0.$$

4. A complete L-function:

$$\Lambda(\varpi, s) = q(\varpi)^{s/2} \gamma(\varpi, s) L(\varpi, s)$$

that is holomorphic in $\Re s > 1$ and is required to admit analytic continuation to a meromorphic function (of order 1) on $\mathbb{C}$ with at most poles at $s = 0$ and $s = 1$.

What is a L-function? (cont.)

5. A functional equation

$$\Lambda(\varpi, s) = \varepsilon(\varpi)\Lambda(\overline{\varpi}, 1-s)$$

where $\varepsilon(\varpi)$ is a complex number of norm 1 and L$(\overline{\varpi}, s)$ is the "dual" of $L(\varpi, s)$:

- $\lambda_{\overline{\varpi}}(n) = \overline{\lambda_{\varpi}(n)}$
- $\gamma(\overline{\varpi}, s) = \gamma(\varpi, s)$
- $q(\overline{\varpi}, s) = q(\varpi, s)$.

Family of L-functions.

- This notion is not well defined in full generalities
- hence we shall use it only in particular cases where it appears "natural".
- In general, at least one parameter is to be taken in account, the conductor: a family $\mathscr{F}_Q$ has a parameter Q such that:

$$L(\varpi, s) \in \mathscr{F}_Q \Longrightarrow q(\varpi) = Q.$$

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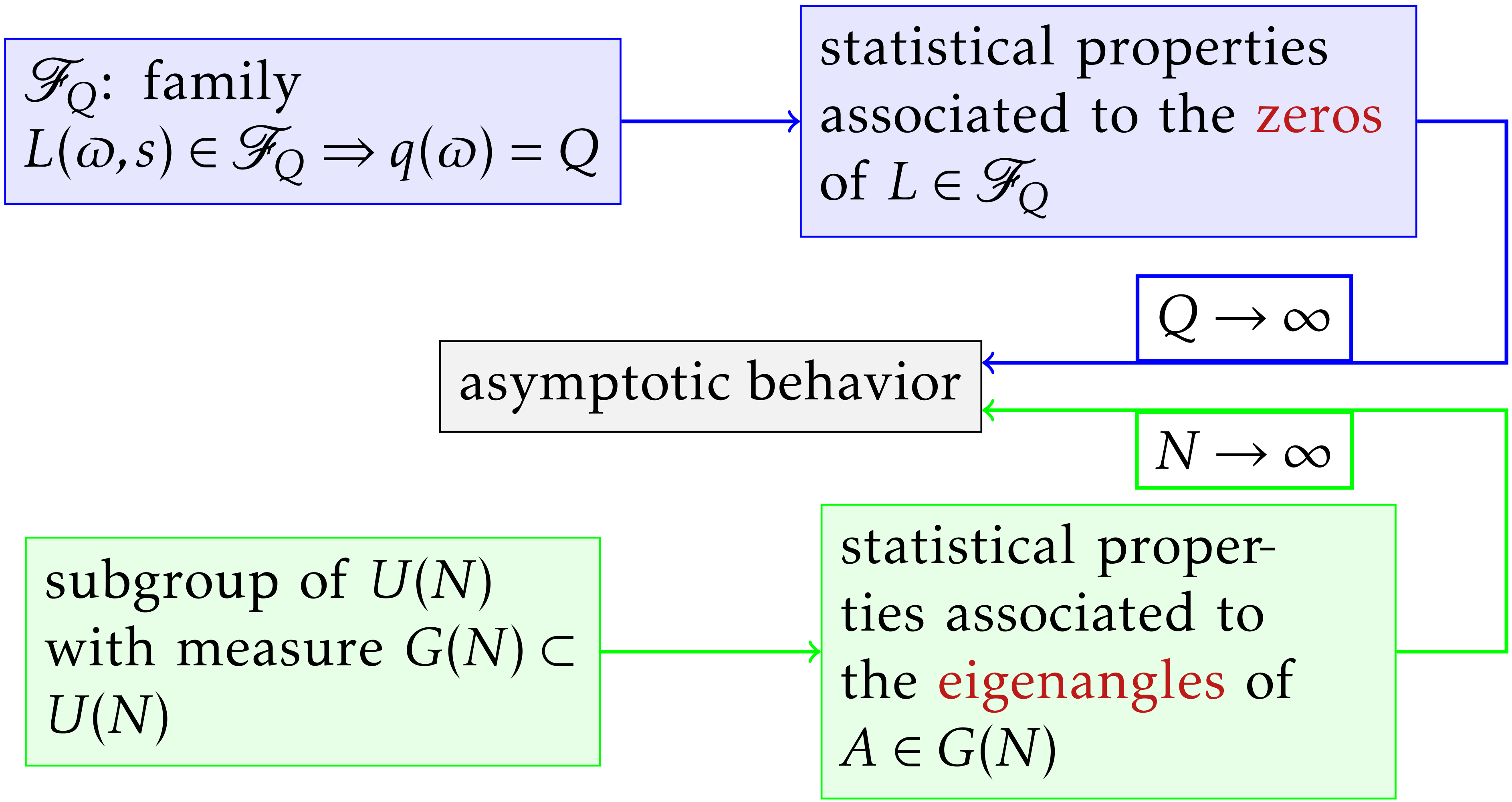
Random matrix theory and L-functions: general view



Parabolic forms

Let $k \geq 2$ be an even integer and $N \geq 1$ a squarefree integer.
A parabolic (modular) form of weight k and level N is a holomorphic function f on the Poincaré upper half-plane $\mathscr{H} = \{z \in \mathbb{C} : \mathfrak{Im} z > 0\}$ such that

1. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : c \mid N \right\}$,

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z)$$

2. the function $z \mapsto (\mathfrak{Im} z)^{k/2} |f(z)|$ is bounded on $\mathscr{H}$.

We get a finite dimensional vector space over $\mathbb{C}$.

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Dirichlet series of parabolic forms

If f is a parabolic form of weight k and level N, it is periodic of period 1 and admits a Fourier expansion

$$f(z) = \sum_{n=1}^{+\infty} \widehat{f}(n) e^{2\pi i n z}.$$

We define its Dirichlet series and completed Dirichlet series by

$$D(f,s) = \sum_{n=1}^{+\infty} \widehat{f}(n) n^{-s}$$

$$\Delta(f,s) = \left(\frac{N}{4\pi^2}\right)^{s/2} \Gamma(s) D(f,s).$$

Dirichlet series of parabolic forms (cont.)

Let $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ and define

$$f|W_N(z) = (\det W_N)^{k/2}(Nz)^{-k} f\left(\frac{-1}{Nz}\right).$$

Since $W_N \Gamma_0(N) W_N^{-1} = \Gamma_0(N)$ this is also a parabolic form of weight k and level N. It is easy seen that

$$\Delta(f,s) = i^k \int_1^{+\infty} f|W_N\left(\frac{it}{\sqrt{N}}\right) t^{k-s}\frac{\mathrm{d}t}{t} + \int_1^{+\infty} f\left(\frac{it}{\sqrt{N}}\right) t^{s}\frac{\mathrm{d}t}{t}$$

hence

$$\Delta(f,s) = i^k \Delta\left(f|W_N, k-s\right).$$

Wanted: Euler product

However these Dirichlet series lack an Euler product (multiplicativity properties for the Fourier coefficients) to be L-functions! To enter the world of L-functions, we need the theory of Atkin & Lehner. It will be possible only on a subspace of parabolic forms, called the subspace of newforms.

Scalar product

On the space of modular forms, we have the Petersson scalar product:

$$(f,g) = \int_{\Gamma_0(N)\backslash\mathscr{H}} f(x+iy)\overline{g(x+iy)}y^k\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

where $\Gamma_0(N)\backslash\mathscr{H}$ is any representative set of the homographic action of $\Gamma_0(N)$ on $\mathscr{H}$:

$$M.z \sim z \Leftrightarrow M \in \Gamma_0(N).$$

Hecke operators

For any integer n, the nth Hecke operator T_n defined by:

$$T_n f(z) = \frac{1}{n} \sum_{\substack{ad=n \\ (a,N)=1}} a^k \sum_{b=0}^{d-1} f\left(\frac{az+b}{cz+d}\right)$$

acts on parabolic forms of weight k and level N. The Hecke operators commute, more precisely they enjoy the following multiplicativity relation:

$$T_m T_n = \sum_{\substack{d|(m,n) \\ (d,N)=1}} d^{k-1} T_{mn/d^2}.$$

Hecke operators (cont.)

Nearly all Hecke operators are selfadjoint: if f and g are parabolic forms of weight k and level N and if n is coprime to N then

$$(T_n f, g) = (f, T_n g).$$

Therefore, we can find an orthogonal basis of the space of parabolic forms of weight k and level N,of eigenvectors of $\{T_n, (n, N) = 1\}$.

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Hecke operators (cont.)

The mth Fourier coefficient of $T_n f$ is given in terms of the Fourier coefficients of f by:

$$\widehat{T_n f}(m) = \sum_{\substack{d|(m,n)\\(d,N)=1}} d^{k-1}\widehat{f}\left(\frac{mn}{d^2}\right)$$

hence, if f is an eigenvector of any T_n with $(n,N)=1$ the eigenvalue $t_f(n)$ satisfies

$$\widehat{f}(n) = \widehat{f}(1)t_f(n).$$

Since we can build parabolic forms $f \neq 0$ with $\widehat{f}(1)=0$ that are eigenvectors of $\{T_n, (n,N)=1\}$ this is not an interesting information.

Atkin-Lehner theory

Define the space of old forms of level N:

$$\mathrm{Old}_k(N) = \mathrm{Vect}\{z \mapsto f(Lz) \colon LM \mid N, M \neq N, f \text{ of level } M\}$$

and the space of newforms:

$$\mathrm{New}_k(N) = \mathrm{Old}_k(N)^{\perp}.$$

Atkin & Lehner theory provides an orthogonal basis of $\mathrm{New}_k(N)$ made of eigenvectors of all the Hecke operators. We can normalize these eigenvectors to have first Fourier coefficient equal to 1. This basis is denoted by $H_k^*(N)$ and its member are called primitive forms

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Primitive forms

If $f \in H_k^*(N)$ then

- f is an eigenvector of T_n with eigenvalue its nth Fourier coefficient (hence this coefficient is real) for any n
- the Fourier coefficients enjoy the same multiplicativity relation as the Hecke operators
- Deligne's theorem

$$|\widehat{f}(n)| \leq \sigma_0(n) n^{(k-1)/2}$$

- there exists $\varepsilon_f(N) \in \{-1, 1\}$ such that $f|W_N = \varepsilon_f(N) f$.

We normalize the Fourier coefficients $\lambda_f(n) = \frac{\widehat{f}(n)}{n^{(k-1)/2}}$.

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L-functions of primitive forms

Let $f \in H_k^*(N)$ and define its L-function by

$$L(f,s) = \sum_{n=1}^{+\infty} \lambda_f(n) n^{-s}.$$

- Euler product:

$$L(f,s) = \prod_{p \in \mathscr{P}} \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}$$

where $\alpha_f(p) + \beta_f(p) = \lambda_f(p)$, $\alpha_f(p) = \beta_f(p)^{-1} = \overline{\beta_f(p)}$ if $(p,N) = 1$ and $\beta_f(p) = 0$ if $p \mid N$.

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L-functions of primitive forms (cont)

- gamma factor

$$\gamma(f,s) = \pi^{-s}\Gamma\left(\frac{s+(k-1)/2}{2}\right)\Gamma\left(\frac{s+(k+1)/2}{2}\right)$$

- conductor N
- the completed L function admits an entire continuation
- functional equation: $\Lambda(f,s) = i^k\varepsilon_f(N)\Lambda(f,1-s)$.

L-functions of primitive forms: zeros

The zeros of $\Lambda(f,s)$ – that is the zeros of $L(f,s)$ that do not compensate the poles of the gamma factor – are inside the critical strip:

$$0 \leq \Re e\, s \leq 1$$

and if ρ is a zero, then $\overline{\rho}$, $1-\rho$ and $1-\overline{\rho}$ also are. To avoid technicalities we assume the Riemann hypothesis for L-functions of primitive forms:

$$\Lambda(f,s) = 0 \Rightarrow \Re e\, s = \frac{1}{2}.$$

If $\Lambda(f, 1/2 + i\gamma_f) = 0$, we associate a normalised zero:

$$\widetilde{\gamma_f} = \frac{\log(k^2N)}{2\pi}\gamma_f.$$

Mean spacing

Let $\mathscr{Z}(f)$ the set imaginary parts of normalised zeros of $\Lambda(f,s)$ (repeated with multiplicities). If $\gamma \in \mathscr{Z}(f)$ is nonnegative, we define the spacing:

$$E(\gamma) = \min\{\gamma - \gamma' : \gamma' \in \mathscr{Z}(f) \setminus \{\gamma\}, 0 \leq \gamma' \leq \gamma\} \quad (\min(\emptyset) = 0).$$

The normalisation of the zeros is chosen as to obtain a unit mean spacing:

$$\lim_{T \to +\infty} \frac{1}{\#\{\gamma \in \mathscr{Z}(f) : 0 \leq \gamma \leq T\}} \sum_{\substack{\gamma \in \mathscr{Z}(f) \\ 0 \leq \gamma \leq T}} E(\gamma) = 1.$$

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One-level density

Let Φ be a Schwartz function whose Fourier transform

$$\widehat{\Phi}(\xi) = \int_{\mathbb{R}} \Phi(x) \exp(-2\pi i x\xi)\, \mathrm{d}x$$

has Fourier compact support. We define the one-level density of $L(f,s)$ by:

$$D_1[\Phi](f) = \sum_{\gamma \in \mathscr{Z}(f)} \Phi(\gamma).$$

Since Φ is of fast decreasing, this evaluates the number of zeros of $\Lambda(f,s)$ in bounded intervals. Due to the normalisation (mean spacing 1) this is expected to be bounded, regardless of the value of N.

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One-level density (cont.)

Analysis is not able to catch only a finite number of zeros of a single L-function. Hence, we study the one-level density on average.

Theorem (Iwaniec, Luo & Sarnak, 2000)

Assume that $\widehat{\Phi}$ *has support in* $(-2,2)$. *Then*

$$\lim_{N\to\infty} \frac{1}{\# H_k^*(N)} \sum_{f\in H_k^*(N)} D_1[\Phi](f) = \int_{\mathbb{R}} \Phi(x) W[O](x)\,\mathrm{d}x$$

with

$$W[O](x) = 1 + \frac{1}{2}\mathrm{Dirac}_0(x).$$

One-level density on RMT side

Let $A \in U(N)$. Its spectrum is

$$\left\{e^{i\varphi_j(A)} \colon 1 \leq j \leq N\right\}$$

for some choice of angles $\{\varphi_j(A) \colon 1 \leq j \leq N\}$. We define its "full anglespectrum":

$$\mathrm{Fas}(A) = \bigcup_{\ell \in \mathbb{Z}} \{\varphi_j(A) + 2\ell\pi \colon 1 \leq j \leq N\}$$

which becomes independant on any choice.

One-level density on RMT side (cont.)

The one level density of $A \in U(N)$ is

$$D_1[\Phi](A) = \sum_{\varphi \in \mathrm{Fas}(A)} \Phi\left(\frac{N}{2\pi}\varphi\right).$$

Theorem (Katz & Sarnak, 1999)

We have

$$\lim_{N\to\infty} \int_{O(N)} D_1[\Phi](A) \,\mathrm{d}\,\mathrm{Haar}_{O(N)}(A) = \int_{\mathbb{R}} \Phi(x) W[O](x) \,\mathrm{d}x$$

with

$$W[O](x) = 1 + \frac{1}{2}\mathrm{Dirac}_0(x).$$

One-level density: smaller families

We can split the set of primitive forms into two smaller sets:

- the set of "even primitive forms":

$$H_k^+(N) = \{f \in H_k^*(N) \colon i^k \varepsilon_f(N) = 1\}$$

- the set of "odd primitive forms":

$$H_k^-(N) = \{f \in H_k^*(N) \colon i^k \varepsilon_f(N) = -1\}$$

These two sets have (asymptotically in N) the same size: half the one of $H_k^*(N)$. Note that, thanks to functional equation:

$$f \in H_k^-(N) \Rightarrow L\left(f, \frac{1}{2}\right) = 0.$$

One-level density: smaller families (cont.)

Theorem (Iwaniec, Luo & Sarnak, 2000)

Assume that $\widehat{\Phi}$ has support in $(-2,2)$. Then

$$\lim_{N\to\infty}\frac{1}{\#H_k^+(N)}\sum_{f\in H_k^+(N)} D_1[\Phi](f)=\int_{\mathbb{R}}\Phi(x)W[SO^+](x)\,\mathrm{d}x$$

with

$$W[SO^+](x)=1+\frac{\sin(2\pi x)}{2\pi x}$$

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One-level density: smaller families (cont.)

Theorem (Iwaniec, Luo & Sarnak, 2000)

Assume that $\widehat{\Phi}$ *has support in* $(-2,2)$. *Then*

$$\lim_{N\to\infty}\frac{1}{\# H_k^-(N)}\sum_{f\in H_k^-(N)} D_1[\Phi](f)=\int_{\mathbb{R}}\Phi(x)W[SO^-](x)\,\mathrm{d}x$$

with

$$W[SO^-](x)=1-\frac{\sin(2\pi x)}{2\pi x}+\mathrm{Dirac}_0(x).$$

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One-level density on RMT side: smaller subgroups

Theorem (Katz & Sarnak, 1999)

We have

$$\lim_{N\to\infty}\int_{SO(2N)} D_1[\Phi](A)\,\mathrm{d}\,\mathrm{Haar}_{SO(2N)}(A) = \int_{\mathbb{R}} \Phi(x) W[SO^+](x)\,\mathrm{d}x$$

with

$$W[SO^+](x) = 1 + \frac{\sin(2\pi x)}{2\pi x}.$$

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One-level density on RMT side: smaller subgroups

Theorem (Katz & Sarnak, 1999)

We have

$$\lim_{N\to\infty}\int_{SO(2N+1)} D_1[\Phi](A)\,\mathrm{d}\,\mathrm{Haar}_{SO(2N+1)}(A) = \int_{\mathbb{R}} \Phi(x) W[SO^-](x)\,\mathrm{d}x$$

with

$$W[SO^-](x) = 1 - \frac{\sin(2\pi x)}{2\pi x} + \mathrm{Dirac}_0(x).$$

Importance of the support of $\widehat{\Phi}$

The model for a family of L-function is provided by one of the three integrals

$$\int_{\mathbb{R}} \Phi W[O], \int_{\mathbb{R}} \Phi W[SO^+], \int_{\mathbb{R}} \Phi W[SO^-]$$

However these integrals equal

$$\int_{\mathbb{R}} \widehat{\Phi}\widehat{W[O]}, \int_{\mathbb{R}} \widehat{\Phi}\widehat{W[SO^+]}, \int_{\mathbb{R}} \widehat{\Phi}\widehat{W[SO^-]}$$

and

$$\widehat{W[O]}|_{(-1,1)} = \widehat{W[SO^-]}|_{(-1,1)} = \widehat{W[SO^+]}|_{(-1,1)}.$$

Even smaller families

If N is squarefree, write $N = p_1 \cdots p_\ell$ with $p_1 < \cdots < p_\ell$ its prime decomposition. We have defined an operator $W_N : f \mapsto f|W_N$. Actually, we have $W_N = W_{p_1} \cdots W_{p_\ell}$ where:

$$f|W_p(z) = p^{k/2}(Nz + pd)^{-k} f\left(\frac{paz + b}{Nz + pd}\right)$$

with

$$(a, b, d) \in \mathbb{Z}^3, d \equiv 1 \pmod{N/p}, p^2 ad - bN = p.$$

If $f \in H_k^*(N)$ then $f|W_{p_j} = \varepsilon_f(p_j) f$ where $\varepsilon_f(p_j) = \pm 1$.

Even smaller families (cont.)

Let $\sigma \in \{-1,1\}^\ell$. We define

$$H_k^\sigma(N) = \Big\{ f \in H_k^*(N) \colon \big(\varepsilon_f(p_1), \dots, \varepsilon_f(p_\ell)\big) = \sigma \Big\}.$$

It can be shown that

$$\# H_k^\sigma(N) \sim \frac{1}{2^\ell} \# H_k^*(N) \quad (N \to +\infty, \omega(N) = \ell)$$

The one-level density of the zeros of L-functions of forms in $H_k^\sigma(N)$ only depends of the sign of the functional equation.

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Even smaller families (cont.)

Theorem (Royer, 2001)

Let ℓ a fixed positive integer and $\kappa \in (0, 1/\ell)$. Let $\boldsymbol{\sigma} \in \{-1, 1\}^\ell$ and Φ whose Fourier transform has compact support in $(-2, 2)$. Consider an infinite sequence $\mathcal{N}$ of squarefree integers having ℓ prime divisors and such that $N^\kappa < \min(p \mid N)$. Then

$$\lim_{\substack{N\to\infty \\ N\in\mathcal{N}}} \frac{1}{\# H_k^{\sigma}(N)} \sum_{f\in H_k^{\sigma}(N)} D_1[\Phi](f) = \int_{\mathbb{R}} W[SO^{\varepsilon}]\Phi(x)\,\mathrm{d}x$$

where $\varepsilon = i^k \sigma_1 \cdots \sigma_\ell$.

Higher moments

To simplify notations, we define three expectation operators for $X\colon H_k^*(N) \to \mathbb{C}$:

$$\mathbb{E}[X] = \frac{1}{H_k^*(N)} \sum_{f \in H_k^*(N)} X(f)$$

$$\mathbb{E}^+[X] = \frac{1}{H_k^+(N)} \sum_{f \in H_k^+(N)} X(f)$$

$$\mathbb{E}^-[X] = \frac{1}{H_k^-(N)} \sum_{f \in H_k^-(N)} X(f)$$

Higher moments (cont.)

Theorem (Hughes & Miller, 2007)

Let $n \geq 2$ and assume that $\widehat{\Phi}$ has compact support in $\left(-\frac{1}{n-1}, \frac{1}{n-1}\right)$. Then

$$\lim_{\substack{N\to\infty \\ N\in\mathscr{P}}} \mathbb{E}^{\pm}\left[\left(D_1[\Phi] - \mathbb{E}^{\pm}\left[D_1[\Phi]\right]\right)^n\right] = \alpha(n) \pm R_n(\Phi)$$

where $\alpha(2m) = (2m-1)!!\sigma_\Phi^{2m}$ and $\alpha(2m+1) = 0$,

$$\sigma_\Phi^2 = 2\int_{-1}^{1} |y|\widehat{\Phi}(y)^2 \,\mathrm{d}y,$$

$$R_n(\Phi) = (-2)^{n-1}\int_{\mathbb{R}} \Phi(x)^n \left[\frac{\sin(2\pi x)}{2\pi x} - \frac{1}{2}\mathrm{Dirac}_0(x)\right]\mathrm{d}x.$$

Higher moments: RMT settings

Exactly the same result holds for subgroups of $U(N)$ when defining for $X\colon U(N)\to\mathbb{C}$:

$$\mathbb{E}^+[X]=\int_{SO(2N)} X(A)\,\mathrm{d}\,\mathrm{Haar}_{SO(2N)}(A)$$
$$\mathbb{E}^-[X]=\int_{SO(2N+1)} X(A)\,\mathrm{d}\,\mathrm{Haar}_{SO(2N+1)}(A).$$

Mock Gaussian behavior

We have:

$$\frac{1}{(-2)^{n-1}} R_n(\Phi) = \int_{\mathbb{R}} \Phi^n S - \frac{1}{2}\Phi(0)^n = \int_{\mathbb{R}} \widehat{\Phi^n}\widehat{S} - \frac{1}{2}\Phi(0)^n$$

with $S(x) = \frac{\sin(2\pi x)}{2\pi x}$. Assume that the support of $\widehat{\Phi}$ is in $\left(-\frac{1}{n}, \frac{1}{n}\right)$ so that the one of $\widehat{\Phi^n}$ is in $(-1, 1)$. Then,

$$\int_{\mathbb{R}} \widehat{\Phi^n}\widehat{S} = \int_{-1}^{1} \widehat{\Phi^n}\widehat{S} = \frac{1}{2}\int_{-1}^{1} \widehat{\Phi^n} = \frac{1}{2}\Phi(0)^n$$

so that

$$R_n(\Phi) = 0.$$

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Mock Gaussian behavior (cont.)

We deduce that the first moments (comparing to the support of $\widehat{\Phi}$ are Gaussian, but that this does not remain for highest moments. This phenomenon, first discovered by Hughes & Rudnick for L-functions of Dirichlet characters is called Mock Gaussian behavior.

Symmetric power L-functions

For any natural integer $r \geq 1$, the symmetric power rth function of associated to $f \in H_k^*(N)$ is the following Euler product:

$$L(\mathrm{Sym}^r f, s) = \prod_{p \in \mathscr{P}} \prod_{j=0}^{r} \left(1 - \frac{\alpha_f(p)^i \beta_f(p)^{r-i}}{p^s}\right)^{-1}.$$

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Symmetric power L-functions: gamma factor

If r is odd, the gamma factor is

$$\gamma(\mathrm{Sym}^r f, s) = \pi^{-(r+1)s/2} \times \prod_{\ell=0}^{(r-1)/2} \Gamma\left(\frac{s+(2\ell+1)(k-1)/2}{2}\right)\Gamma\left(\frac{s+1+(2\ell+1)(k-1)/2}{2}\right)$$

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Symmetric power L-functions: gamma factor

If r is even, the gamma factor is

$$\gamma(\mathrm{Sym}^r f, s) = \pi^{-(r+1)s/2} \times$$
$$\Gamma\left(\frac{s+\mu_{k,r}}{2}\right) \prod_{\ell=1}^{r/2} \Gamma\left(\frac{s+\ell(k-1)}{2}\right) \Gamma\left(\frac{s+1+\ell(k-1)}{2}\right)$$

with

$$\mu_{k,r} = \begin{cases} 1 & \text{if } r(k-1)/2 \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

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Symmetric power L-functions: functional equation

The conductor is N^r and the functional equation is

$$\Lambda(\mathrm{Sym}^r f, s) = \varepsilon_{\mathrm{Sym}^r f}(N)\Lambda(\mathrm{Sym}^r f, s)$$

where $\varepsilon_{\mathrm{Sym}^r f}(N)$ is $\varepsilon_f(N)$ up to a sign depending only on the fixed variables k and r. This symmetric power L-function admits an entire continuation.
The functional equation and continuation are known if $1 \leq r \leq 4$ (Hecke, Gelbart & Jacquet, Kim & Shahidi) and conjectural for $r > 4$. This is a consequence of the Langlands modularity conjecture.

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Symmetric power L-functions: symmetry type

We want to determine which subgroup of $U(N)$ can be used to modelise the zeros of symmetric power L-functions. We assume Riemann hypothesis for these L-functions and define the one-level density by

$$D_1[\Phi; r](f) = \sum_{\gamma,\ \Lambda(\mathrm{Sym}^r f, 1/2+i\gamma)=0} \Phi\left(\frac{\log(N^r)}{2\pi}\gamma\right)$$

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Symmetric power L-functions: symmetry type

Theorem (Ricotta & Royer, 2007)

Assume $\widehat{\Phi}$ has support in $(-\nu,\nu)$. Let $\theta = 7/64$ and

$$\nu_{1,\max}(r,\kappa,\theta) := \left(1 - \frac{1}{2(\kappa-2\theta)}\right)\frac{2}{r^2}.$$

If $\nu < \nu_{1,\max}(r,\kappa,\theta)$ then the asymptotic expectation of the one-level density is

$$\widehat{\Phi}(0) + \frac{(-1)^{r+1}}{2}\Phi(0).$$

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Symmetric power L-functions: symmetry type

It follows that:

- if r is even, the zeros are modelised by matrices in Sp
- if $r = 1$ the zeros are modelised by matrices in O
- if $r \geq 3$ is odd, the zeros are modelised by matrices in O or SO^+ or SO^-.

The support is too small to determine the symmetry type in the case of odd $r \geq 3$. We shall use the two level density.

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Numbering of the zeros

The set of the zeros of $\Lambda(\mathrm{Sym}^r f, s)$ counted with multiplicities is given by

$$\left\{\frac{1}{2} + i\gamma_{f,r}^{(j)} : j \in \mathscr{E}(f,r)\right\}$$

where

$$\mathscr{E}(f,r) := \begin{cases} \mathbb{Z} & \text{if } \varepsilon(\mathrm{Sym}^r f) = -1 \\ \mathbb{Z} \setminus \{0\} & \text{if } \varepsilon(\mathrm{Sym}^r f) = 1 \end{cases}$$

Numbering of the zeros (cont.)

We enumerate the zeros such that

1. the sequence $j \mapsto \gamma_{f,r}^{(j)}$ is increasing
2. we have $j \geq 0$ if and only if $\gamma_{f,r}^{(j)} \geq 0$
3. we have $\gamma_{f,r}^{(-j)} = -\gamma_{f,r}^{(j)}$.

Numbering of the zeros (cont.)

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Remember that if $\varepsilon(\mathrm{Sym}^r f) = -1$ then the functional equation of $\Lambda(\mathrm{Sym}^r f, s)$ evaluated at the critical point $s = 1/2$ provides a zero denoted by $\frac{1}{2} + i\gamma_{f,r}^{(0)}$.

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Two level density

The two-level density of $\mathrm{Sym}^r f$ (relatively to Φ_1 and Φ_2) is defined by

$$D_2[\Phi_1,\Phi_2;r](f) = \sum_{\substack{(j_1,j_2)\in\mathscr{E}(f,r)^2 \\ j_1\neq\pm j_2}} \Phi_1\left(\frac{\log N^r}{2\pi}\gamma_{f,r}^{(j_1)}\right)\Phi_2\left(\frac{\log N^r}{2\pi}\gamma_{f,r}^{(j_2)}\right).$$

It has been shown by Miller that, this statistics for subgroups of $U(N)$ allow to distinguish the subgroups, regardless to the support of Φ_1 and Φ_2. Hence, it should allow us to determine the symmetry type of our symmetric L-functions.

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Two level density

Theorem (Ricotta & Royer, 2007)

Assume $\widehat{\Phi_1}$ and $\widehat{\Phi_2}$ have support in $(-v, v)$. If $v < 1/r^2$ then the asymptotic expectation of the two-level density is

$$\left[\widehat{\Phi_1}(0) + \frac{(-1)^{r+1}}{2}\Phi_1(0)\right]\left[\widehat{\Phi_2}(0) + \frac{(-1)^{r+1}}{2}\Phi_2(0)\right] + 2\int_{\mathbb{R}} |u|\widehat{\Phi_1}(u)\widehat{\Phi_2}(u)\,\mathrm{d}u - 2\widehat{\Phi_1\Phi_2}(0) + \left((-1)^r + \frac{\chi_{2\mathbb{N}+1}(r)}{2}\right)\Phi_1(0)\Phi_2(0).$$

Symmetric power L-functions: symmetry type

It follows that:

- if r is even, the zeros are modelised by matrices in Sp
- if r is odd, the zeros are modelised by matrices in O.

Symmetric power L-functions: moments

Finally, we are able to compute the moments of the one-level density. However, not in a large enough range to exhibit the Mock Gaussian behavior.

Theorem (Ricotta & Royer, 2007)

Assume $\widehat{\Phi}$ has support in $(-\nu,\nu)$. If $m\nu < 4/(r(r+2))$ then the asymptotic m-th moment of the one-level density is

$$\begin{cases} 0 & \text{if } m \text{ is odd,} \\ \left(2\int_{\mathbb{R}} |u|\widehat{\Phi}^2(u)\,\mathrm{d}u\right)^{m/2} \times \frac{m!}{2^{m/2}\left(\frac{m}{2}\right)!} & \text{otherwise.} \end{cases}$$